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Introduction

The dielectric coefficient of germanium has been determined previously by the optical method[1] and the conductivities of germanium as a function of temperature and impurity contents are measured directly by d. c. measurements.[2] It is the purpose of the present study to apply the microwave technique, developed for the study of problems in gas discharges, to determine the dielectric coefficient as well as the conductivity of germanium. The method is general, however, and may be used for any material of arbitrary electrical properties. The present report concerns only the theoretical aspect of the problem and the experimental part will be reported separately.

It is known that the solutions of Maxwell equations in a hollow cavity can be expressed as a summation of normal modes corresponding to various frequencies. For practical purposes, it is usually convenient to choose the geometry of the cavity so that only one mode exists in the cavity. If the volume of the cavity so designed is perturbed by the insertion of any dielectric substance in part of the cavity, the resonant frequency will differ from that of the empty cavity; moreover, the presence of the inserted substance will introduce losses in addition to the loss due to the walls. From the definition of the Q of a cavity, it can be shown that

$$Q = \frac{1}{2}\frac{\text{Real part of frequency}}{\text{Imaginary part of frequency}} = \frac{1}{2}\frac{\omega_r}{\omega_j} = \frac{1}{2}\cot\phi \tag{1}$$

The electric and magnetic fields depend on time in the form of $e^{j\omega t}$. The purpose of the present method is to determine the dielectric constant and the conductivity from the knowledge of frequency shift and the change of Q.

The concept of a complex frequency is meaningful only in the transient case and it is this case which we consider. Experimental measurements, of course, are made in the steady state. Whenever a resonant circuit or cavity contains losses the transient decay frequency and the steady state resonant frequency are in general different, the relative difference $\Delta\omega/\omega$ being some function of $1/Q^2$. For lumped circuits the relation between the two frequencies depends on the geometry of the circuit. The microwave case is more complicated and one must solve Maxwell's equation for the two cases. The steady state solution is exceedingly difficult for cavities and has not been obtained.* As a result, one must exert caution when applying the results of this paper to steady state measurements when low Q's characterize the resonance.

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*An extension of the solution to the steady case has recently been obtained. Expected publication in Journal of Applied Physics.

Cylindrical Sample as a Center Post of a Cylindrical Cavity

This is the method used by Birnbaum[3] and many others in measuring the dielectric coefficient for low loss dielectric material. The electrical properties of the sample are determined in terms of the frequency shift and change in Q by means of a perturbation method developed by Bethe and Schwinger.[4] The results are as follows:

$$\frac{\lambda_1 - \lambda_0}{\lambda_0} = \frac{\left(\frac{\epsilon_r}{\epsilon_0} - 1\right) \int_{sample} E^2 \, dv}{2 \int_{cavity} E^2 \, dv} \tag{2a}$$

$$\frac{1}{Q_1} - \frac{1}{Q_0} = \frac{\left(\frac{\epsilon_j}{\epsilon_0}\right) \int_{sample} E^2 \, dv}{\int_{cavity} E^2 \, dv} \tag{2b}$$

where λ_0 and λ_1 are the resonant wave lengths for the empty and loaded cavity, respectively. Q_0 and Q_1 represent the unloaded Q's of the empty cavity and the one with sample. The effective dielectric coefficient is defined as $\kappa = \epsilon_r/\epsilon_0$, and the loss tangent is $\tan \delta = \epsilon_j/\epsilon_r$ where ϵ_0 is the electric permittivity in vacuum. The complex dielectric coefficient can be defined as $\epsilon_c = \epsilon_r - j\epsilon_j$. Eqs. (2a) and (2b) are applicable to low loss samples of such dimensions which cause only small frequency shifts. Eq. (2a) gives the effective dielectric coefficient as a linear function of the frequency shift.

In order to get a more general solution, one can solve Maxwell's equations and determine the natural frequency by the boundary conditions. The equations to be solved are:

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \tag{3a}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \sigma \vec{E} \tag{3b}$$

$$\nabla \cdot \vec{E} = 0 \tag{3c}$$

$$\nabla \cdot \vec{H} = 0 \tag{3d}$$

If both the electric and magnetic fields are assumed to have a time dependence of the form $\vec{E} = \vec{E}e^{j\omega t}$, $\vec{H} = \vec{H}e^{j\omega t}$, then the above equations can be simplified to

$$\nabla \times \vec{E} = -j\mu_0 \omega \vec{H} \tag{4a}$$

$$\nabla \times \vec{H} = j\omega(\epsilon_r - j\epsilon_j)\,\vec{E} \tag{4b}$$

$$\nabla \cdot \vec{E} = 0 \tag{4c}$$

$$\nabla \cdot \vec{H} = 0 \tag{4d}$$

where $\sigma = \epsilon_j \omega$ and $\omega = \omega_r + j\omega_j$. Eqs. (4a) to (4d) are to be solved in the region between the sample and the cavity as well as the region inside the sample. The frequency is then determined by the conditions of continuity of both the electric field and magnetic field. The vector wave equations to be solved in the two regions are

$$\nabla \times \nabla \times \vec{E} - k^2\vec{E} = 0 \tag{5}$$

and

$$\nabla \times \nabla \times \vec{E} - k'^2\vec{E} = 0 \tag{6}$$

where

$$k^2 = \mu_0 \epsilon_0 \omega^2$$

$$k'^2 = \mu_0(\epsilon_r - j\epsilon_j)\,\omega^2 = \kappa(1 - j\tan\delta)\,k^2$$

Due to the fact that ω is a complex quantity, it is exceedingly difficult to solve Eqs. (5) and (6) for the most general case. However, by a suitable choice of cavity dimensions one can regard the cavity as oscillating in a particular mode, which is chosen as the TM_{010} mode in the present case. The electric field is parallel to the surface of the sample. It has been shown by Bethe and Schwinger[4] that if the electric field is perpendicular to the sample, then the frequency shift varies as $(\kappa - 1)/\kappa$, which is almost equal to unity for large values of κ. Consequently if a cylindrical germanium sample placed at the center of the cavity is chosen as the geometry for the measurement, then a TM mode should be used. We shall assume (1) that there is only one mode of oscillation, and (2) that the ends of the samples are in good contact with the metal walls of the cavity. This assumption is justified if the sample is soldered to the cavity at both ends. Under the given assumptions, the solutions of Eq. (5) can be written as

$$\vec{E}_z = A\left[J_0(kr) - \frac{J_0(ka)}{N_0(ka)}\,N_0(kr)\right] \tag{7}$$

$$\vec{H}_\phi = jA\sqrt{\frac{\epsilon_0}{\mu_0}}\left[J_1(kr) - \frac{J_0(ka)}{N_0(ka)}\,N_1(kr)\right] \tag{8}$$

where a is the radius of the cavity. The ratio at the surface of the sample ($r = b$) is

$$\frac{H_\phi}{E_z} = j\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{J_1(kb)\,N_0(ka) - J_0(ka)\,N_1(kb)}{J_0(kb)\,N_0(ka) - J_0(ka)\,N_0(kb)} \tag{9}$$

The solutions inside the sample are:

$$\vec{E}_z = BJ_0(k'r) \tag{10}$$

$$\vec{H}_\phi = Bj\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{k'}{k}\,J_1(k'r) \tag{11}$$

and the ratio at the surface is

$$\frac{H_\phi}{E_z} = j\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{k'}{k}\,\frac{J_1(k'b)}{J_0(k'b)} \tag{12}$$

from the condition of continuity, at $r = b$, the following transcendental equation is obtained

$$\kappa^{1/2}(1 - j\tan\delta)^{1/2}\,\frac{J_1(k'b)}{J_0(k'b)} = \frac{J_1(kb)\,N_0(ka) - J_0(ka)\,N_1(kb)}{J_0(kb)\,N_0(ka) - J_0(ka)\,N_0(kb)} \tag{13}$$

The solution of this equation gives the physical properties of interest in terms of measured quantities. In addition to the general solution which is given below, one may obtain a special solution by neglecting the imaginary part of the left-hand side of Eq. (13). This gives a real frequency and is the case to which low loss samples belong. The relation between the effective dielectric coefficient and the resonant frequency can be obtained by solving the transcendental equation (13). Some of the results and the calculation of Q have been obtained by Feenberg.[5] However, the region of interest for the present study is different from that given there.

For samples with very small radii such that: (1) the approximate formulae for Bessel functions of small arguments can be used, and (2) the frequency shift is small such that the values of $J_0(ka)$ and $N_0(ka)$ can be expanded by Taylor's series from the frequency of the empty cavity; the following simplifications can be made:

$$J_0(kb) = 1$$

$$J_1(kb) = \frac{kb}{2}$$

$$N_1(kb) = -\frac{2}{\pi kb}$$

$$N_0(kb) = -\frac{2}{\pi} \ln \frac{2}{kb\gamma}, \quad \gamma \text{ is Euler's constant}$$

$$J_0(ka) = -J_1(k_0a)(k - k_0)a, \quad \text{where } k_0 \text{ satisfies the condition that } J_0(k_0a) = 0$$

$$N_0(ka) = N_0(k_0a) - N_1(k_0a)(k - k_0)a.$$

Using these results, Eq. (13) can be written in the form

$$\kappa = 1 + 0.224\left(\frac{a^2}{b^2}\right)(k_0 - k)a \tag{14}$$

which is the same as that given by the perturbation method. The conductivity in this case can be calculated from the change of Q. Let Q_0 be the Q for the empty cavity and Q be that for the loaded cavity. In general, the value of Q and Q_0 can be expressed as

$$\frac{1}{Q} = \frac{1}{Q_{dielectric}} + \frac{1}{Q_0{}'_{wall}} \tag{15}$$

$$Q_0 = \frac{a}{\delta_0} \frac{1}{1 + \frac{a}{L}} \tag{16}$$

respectively, where δ_0 is the skin depth and L is the height of the cavity. Due to the presence of the dielectric sample, Q_0' differs from Q_0 and can be calculated by using the equation

$$Q_0' = \frac{2}{\delta'} \frac{\int |H|^2 \, d\tau}{\int |H|^2 \, dS} \tag{17}$$

where δ' is the skin depth corresponding to the new frequency. Using the value of H_ϕ and the continuity conditions on the surface of the dielectric, it can be shown that

$$Q = Q_0 \sqrt{\frac{k_0 a}{ka}} \; \frac{1 + \frac{a}{L}}{\dfrac{y^2}{1 + y^2 - \dfrac{k_0^2}{k^2}} + \frac{a}{L}} \tag{18}$$

where

$$y = \frac{2}{\pi} \frac{1}{\sqrt{\kappa}\ kb\ N_0(ka)\ J_0(kb) \left[1 - \frac{J_0(ka)\ N_0(kb)}{N_0(ka)\ J_0(kb)}\right]} .$$

Calculations show that the factor $y^2/\left[1 + y^2 - (k_0{}^2/k^2)\right]$ does not differ much from unity. In general, Q_0' is of the same order of magnitude as Q_0 and can be considered as equal for most cases. The relation between $Q_{dielectric}$ and the loss tangent can then be written as

$$(\tan \delta)\ Q_\varepsilon = \frac{1 + y^2 - \frac{1}{\varepsilon_r}}{1 + \left[\frac{J_1(kb)}{J_0(kb)}\right]^2} . \tag{19}$$

For a sample with low conductivity such that the imaginary part of the frequency can be neglected, Eq. (13) yields the same result as that calculated by the perturbation method.

In Fig. 1 the effective dielectric coefficient calculated from Eq. (14) is plotted as a function of ka for several sample sizes. For sample sizes not shown on the graph κ can be determined by means of a quadratic interpolation.

Fig. 2 gives the values of Q_ε tan δ as a function of ka for a value of 100 for a/b. Having found the value of Q_ε and κ, the conductivity of the sample can be determined. The detailed procedures of the calculations can be outlined as follows:

(1) From the wavelength of the empty cavity λ_0 and that of the loaded cavity λ, the value of ka can be determined by the relation $2.4048/ka = \lambda/\lambda_0$.

(2) Knowing ka, the value of κ can be determined for a given a/b.

(3) From the measured value of Q_ε and Fig. 2, find tan δ. Then it follows that the conductivity is

$$\sigma = \frac{\kappa \tan \delta}{\lambda} \times 1.67 \times 10^{-2} \text{ mho/m}, \quad \lambda \text{ in meters}$$

We next consider the general solution of Eq. (13). In order to solve this transcendental equation the frequency must be complex. The solution has been obtained and is presented graphically below. The natural frequency depends upon both the conducting and dielectric properties of the sample. For small values of the conductivity the frequency is less than that of the empty resonator, and is determined primarily by the dielectric properties of the post. As the conductivity is increased, the natural frequency of the loaded cavity increases. It passes through that of the empty

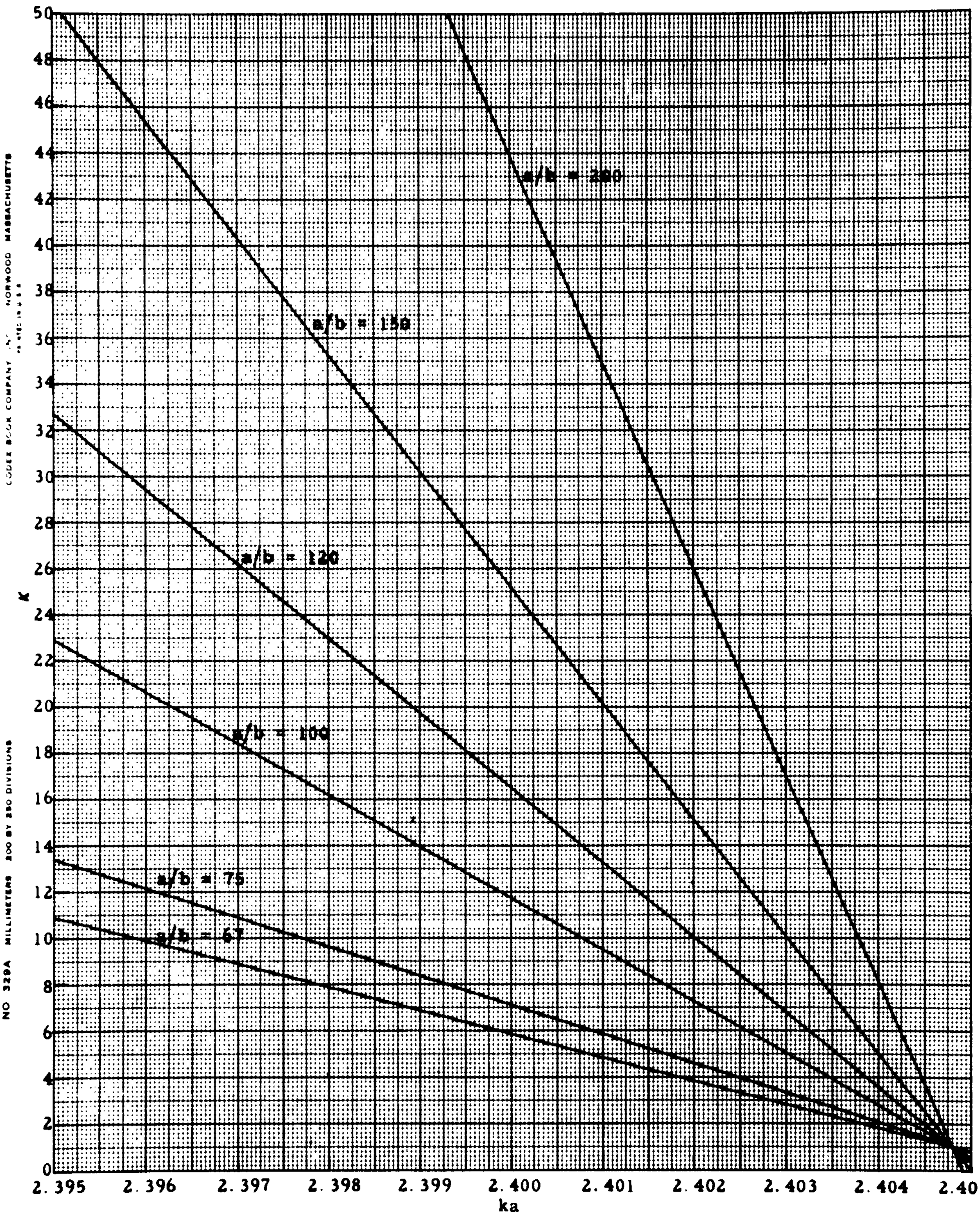

Fig. 1

κ vs ka obtained from Eq. (14)

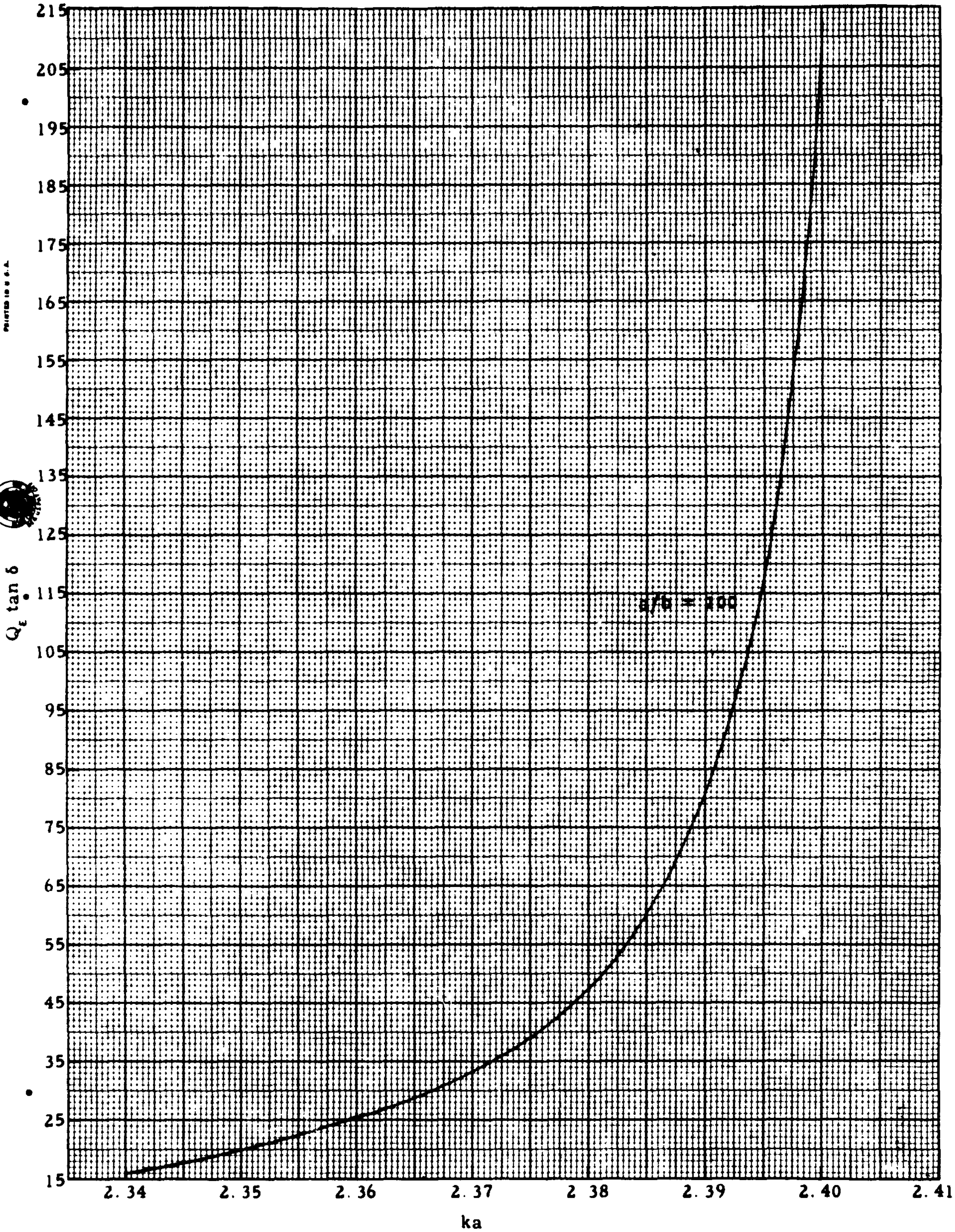

Fig. 2

Q_ε tan δ vs ka for a/b = 100, obtained from Eq. (19)

cavity and asymptotically approaches the resonant frequency of a coaxial cavity as the conductivity becomes infinite. This transition from a perturbed cylindrical cavity to a coaxial cavity is shown in Fig. 3. There we plot natural frequency as a function of $\kappa \tan \delta$ which is proportional to sample conductivity for a given sample size and given value of κ.

This mode transition is also reflected in the loss factor or Q_ε of the cavity. For zero conductivity the Q_ε is infinite of course. As soon as a finite conductivity is assigned losses are introduced. The Q is finite and falls, as the conductivity increases, to a value of about 5. Further increase in the conductivity increases the Q. This is due to the fact that the conduction electrons in the sample now shield the field and prevent it from penetrating the sample. As the conductivity goes infinite, the Q_ε does also. The behavior of the Q_ε as a function of $\kappa \tan \delta$ for a given sample size is shown in Fig. 4.

For a definite frequency, solutions with negative values of κ can also be obtained. This can be understood by rewriting the expression of the dielectric constant using the relations

$$\varepsilon_j = \frac{\sigma}{\omega}$$

and

$$\sigma = \sigma_r + j\sigma_j \tag{20}$$

where σ_j is in general a negative quantity. Then

$$\kappa(1 - j \tan \delta) = \left(\frac{\varepsilon_r}{\varepsilon_0} - j\frac{\varepsilon_j}{\varepsilon_0}\right) = \kappa_e + \frac{\sigma_j}{\omega\varepsilon_0} - j\frac{\sigma_r}{\omega\varepsilon_0} = \kappa - j\frac{\sigma_r}{\omega\varepsilon_0} \tag{21}$$

The real quantity of Eq. (21) can be called the effective dielectric coefficient and this is the quantity that is measured experimentally. The case of a perfect conductor corresponds to a case of infinite effective dielectric constant with a negative sign. As the sample becomes more conducting, the measured frequency should give a negative effective dielectric constant.

Fig. 5 is a plot of the complex permittivity relative to that of free space in the ka plane. The constant κ contours differ from each other at the low frequency end by the amount given by the perturbation theory. At the coaxial limit the curves meet, for here only the conduction electrons are important. The constant $\kappa \tan \delta$ curves all converge toward the point (2.801 + j0) but do not actually reach this point. This is because the imaginary part of ka cannot be zero (an infinite Q) for a finite conductivity. The point however does represent the complete contour for the limiting

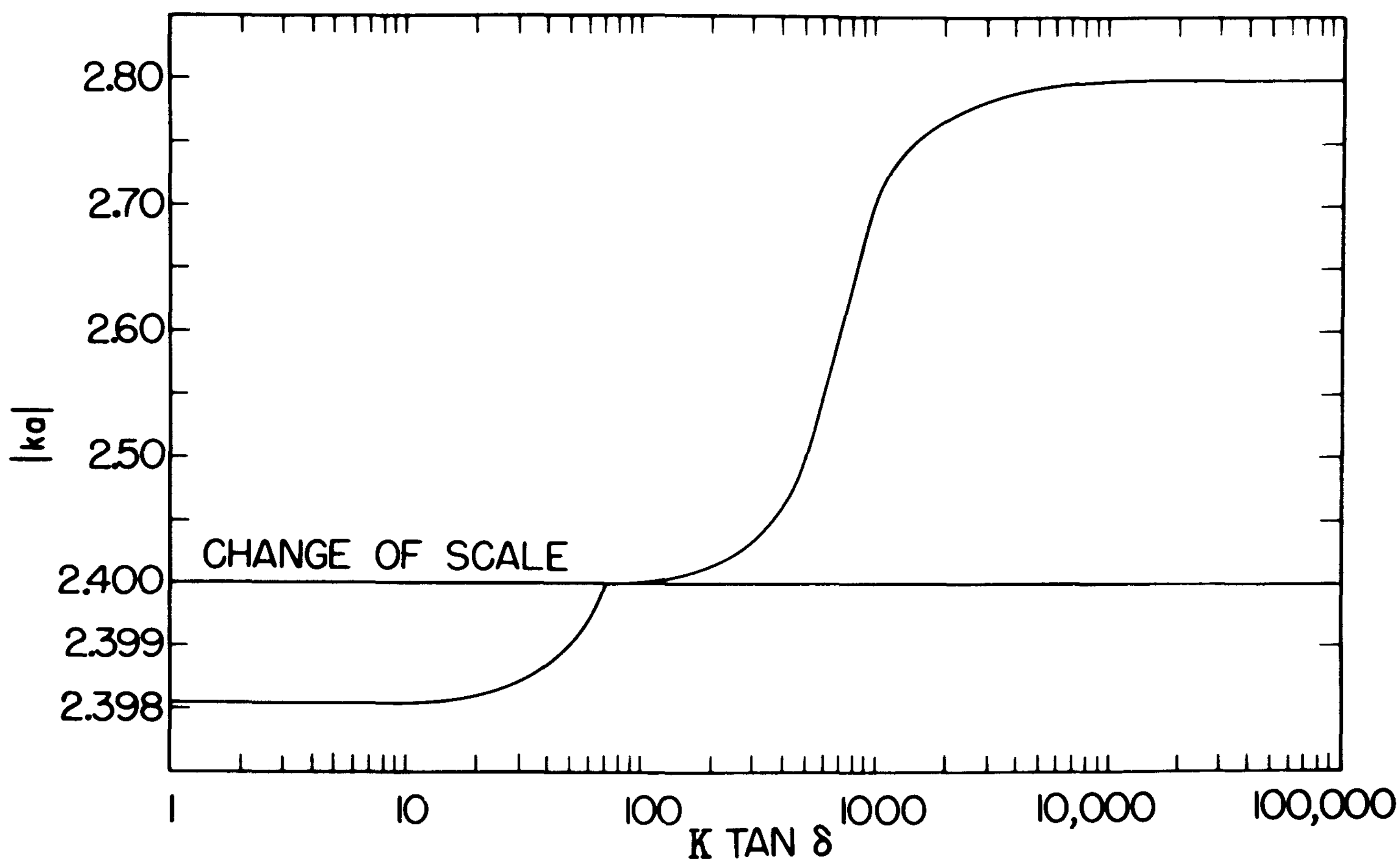

Fig. 3

$|ka|$ vs $\kappa \tan \delta$ for $\kappa = 16$ and $a/b = 100$. This figure shows the transition from the cylindrical mode to the coaxial mode.

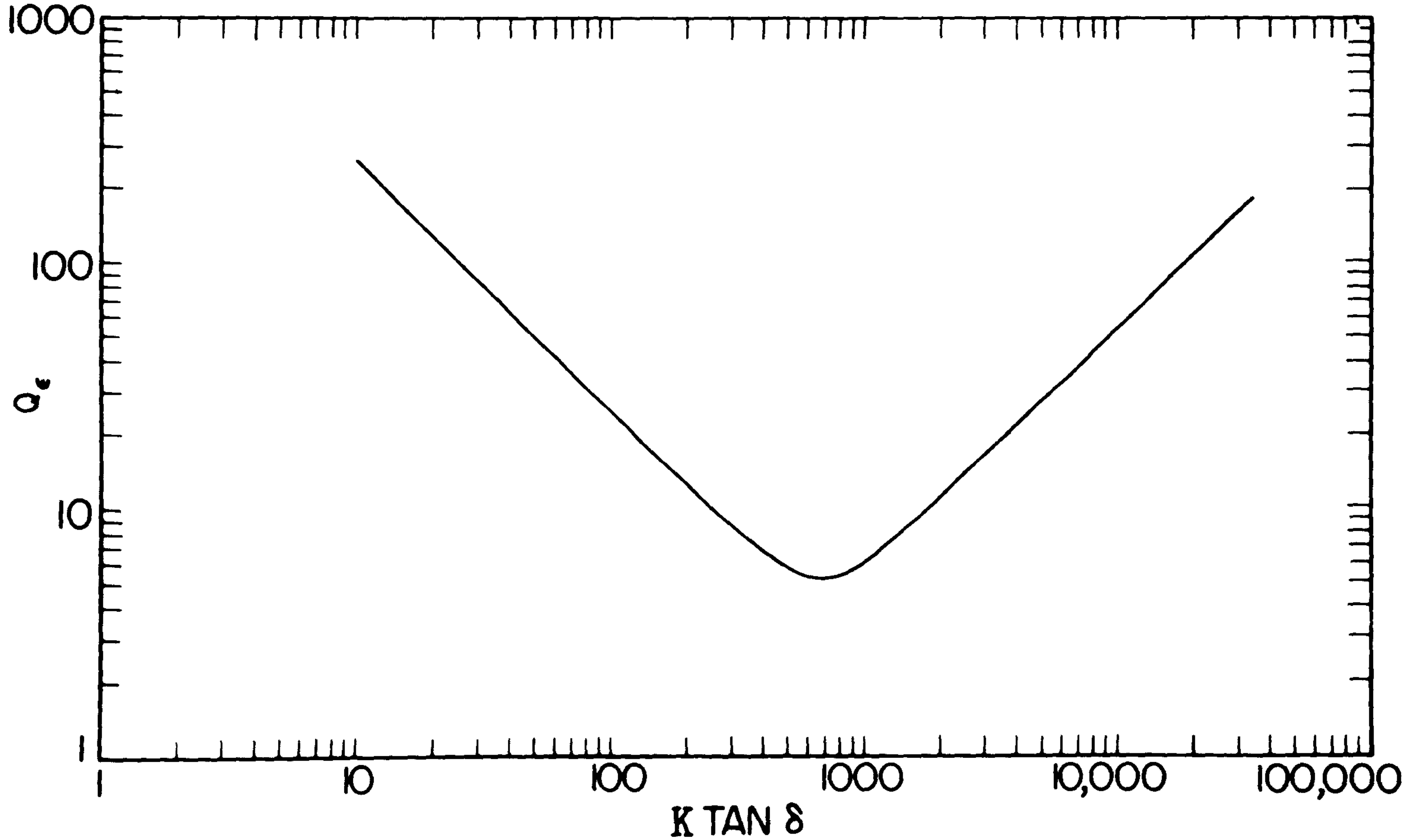

Fig. 4

Q_ε vs κ tan δ for κ = 16 and a/b = 100. Q_ε first decreases as the conductivity increases, but later increases with the conductivity.

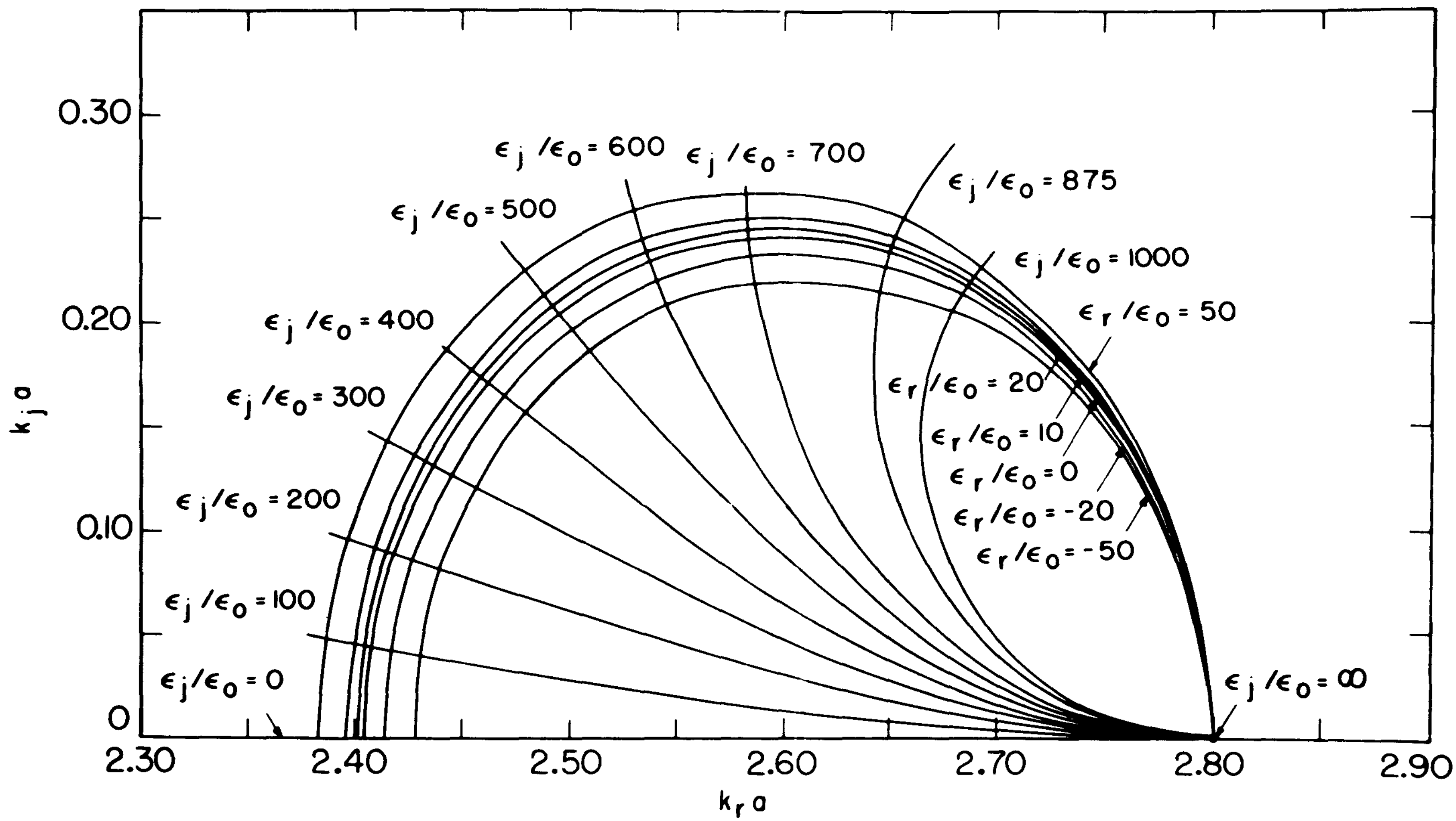

Fig. 5

Map of the complex dielectric coefficient in the ka plane for a/b = 100.

case of $\kappa \tan \delta = \infty$. The other limiting case, $\kappa \tan \delta = 0$ is the positive real axis below 2.4048. On this axis κ is equal to the dielectric coefficient.

Although the solution has been obtained for the complete range of frequencies only that corresponding to negative frequency shifts is given. For positive frequency shifts the Q_ϵ's are so low that accurate experimental measurements are impossible and that part of the solution is of academic interest only.

In Figs. 6-11 we show κ as a function of ϕ for various values of the magnitude of ka. Linear interpolation may be used for those values of ka not shown. This set of figures shows several interesting features. It is seen that the results obtained previously by neglecting the imaginary part are good within 0.3 in κ for tan δ as high as 1.0. In addition the curves show that the frequency shift is a linear function of κ for all values of ϕ.

Figs. 12 - 17 show $\kappa \tan \delta$ as a function of ϕ for various values of the magnitudes of ka. We see that for low ϕ's the frequency is independent of σ, as we have also seen above, but that as σ increases ka is dependent upon it.

Spherical Sample in a Cylindrical Resonant Cavity

The effect of a metallic sphere in a resonant cavity has been considered by Maier and Slater[6] using the perturbation method. It is assumed that the sphere is placed in a constant electrostatic or magnetostatic field. This is not a bad assumption if the diameter is smaller than the wavelength. Similar calculations are carried out for a dielectric sphere of complex dielectric constant. For a dielectric sphere in a uniform field E_0, the potential outside the sphere can be written as[7]

$$\Phi = - E_0 r P_1(\cos \theta) + \frac{\epsilon_r - j\epsilon_j - \epsilon_0}{2\epsilon_0 + \epsilon_r - j\epsilon_j} \frac{b^3}{r^2} E_0 P_1(\cos \theta) \tag{22}$$

where b is the radius of the sphere, $\epsilon_c = \epsilon_r - j\epsilon_j$, and E_0 is the electric field at the center of the cavity. The electric fields E_r, E_θ, and E_ϕ are

$$E_r = \left[1 + \frac{2(\epsilon_c - \epsilon_0)}{2\epsilon_0 + \epsilon_c} \frac{b^3}{r^3}\right] E_0 \cos \theta \tag{23}$$

$$E_\theta = - \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = \left[- 1 + \frac{\epsilon_c - \epsilon_0}{2\epsilon_0 + \epsilon_c} \frac{b^3}{r^3}\right] E_0 \sin \theta \tag{24}$$

$$E_\phi = 0 \tag{25}$$

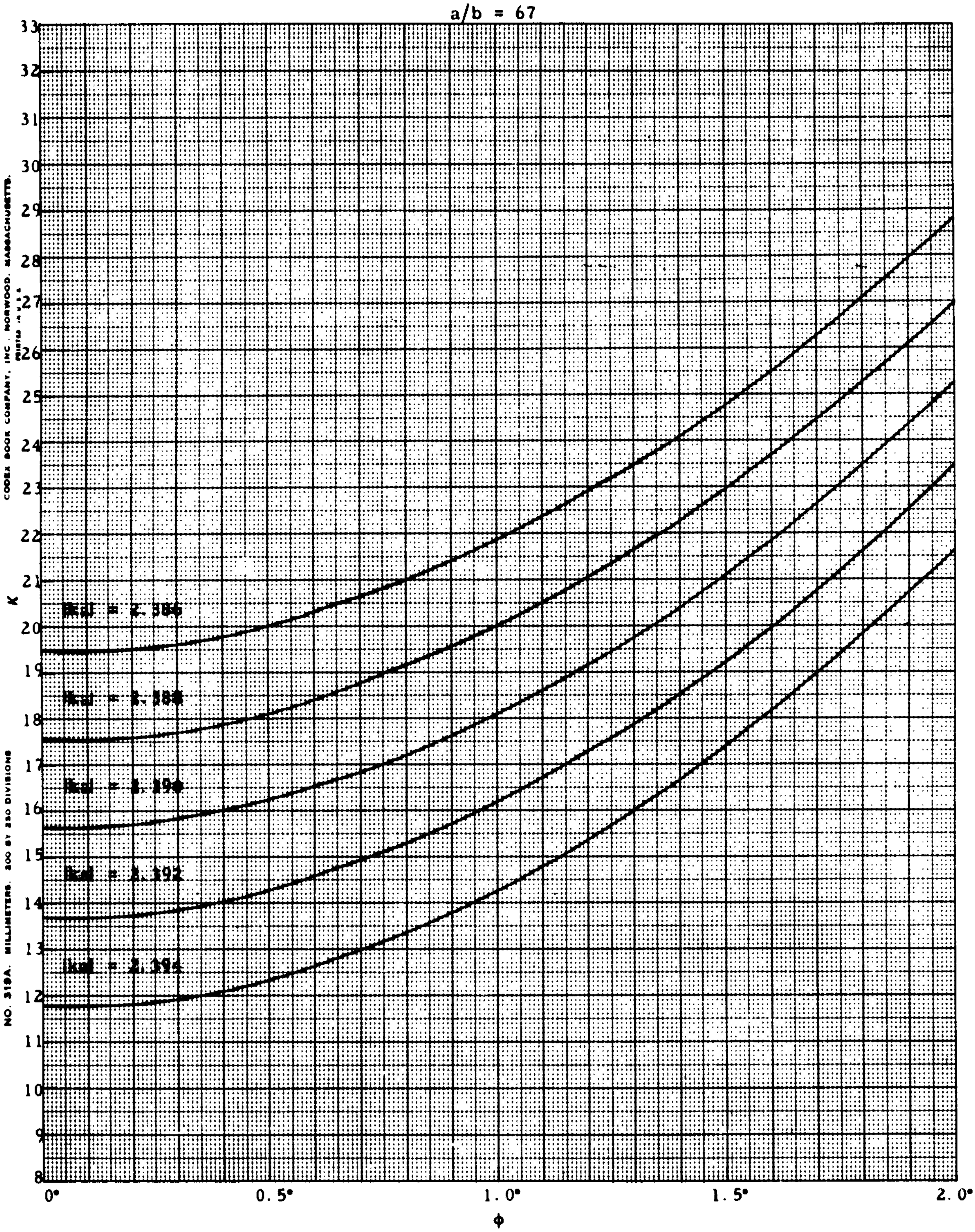

Fig. 6

κ vs ϕ for a/b = 67

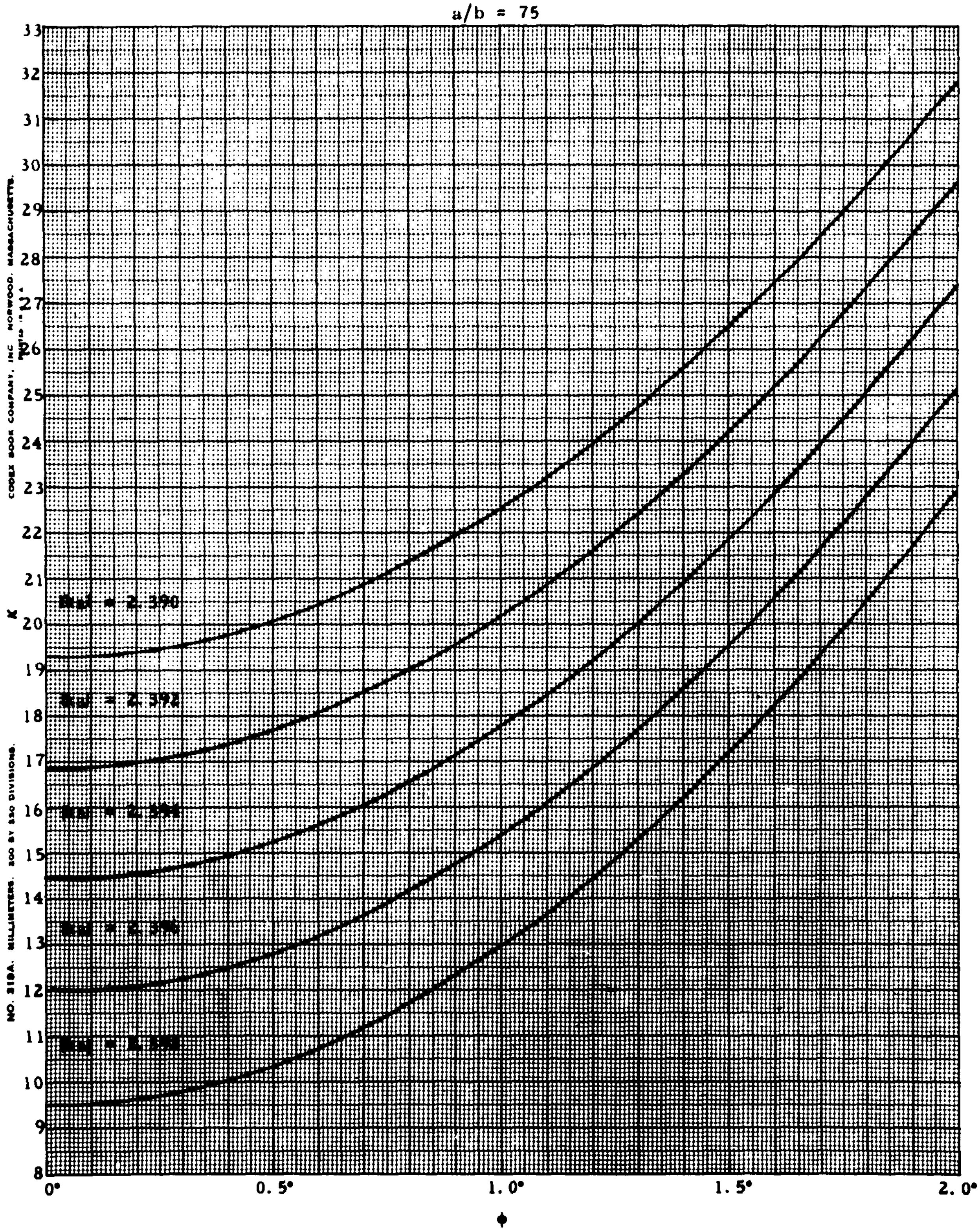

Fig. 7
κ vs ϕ for a/b = 75

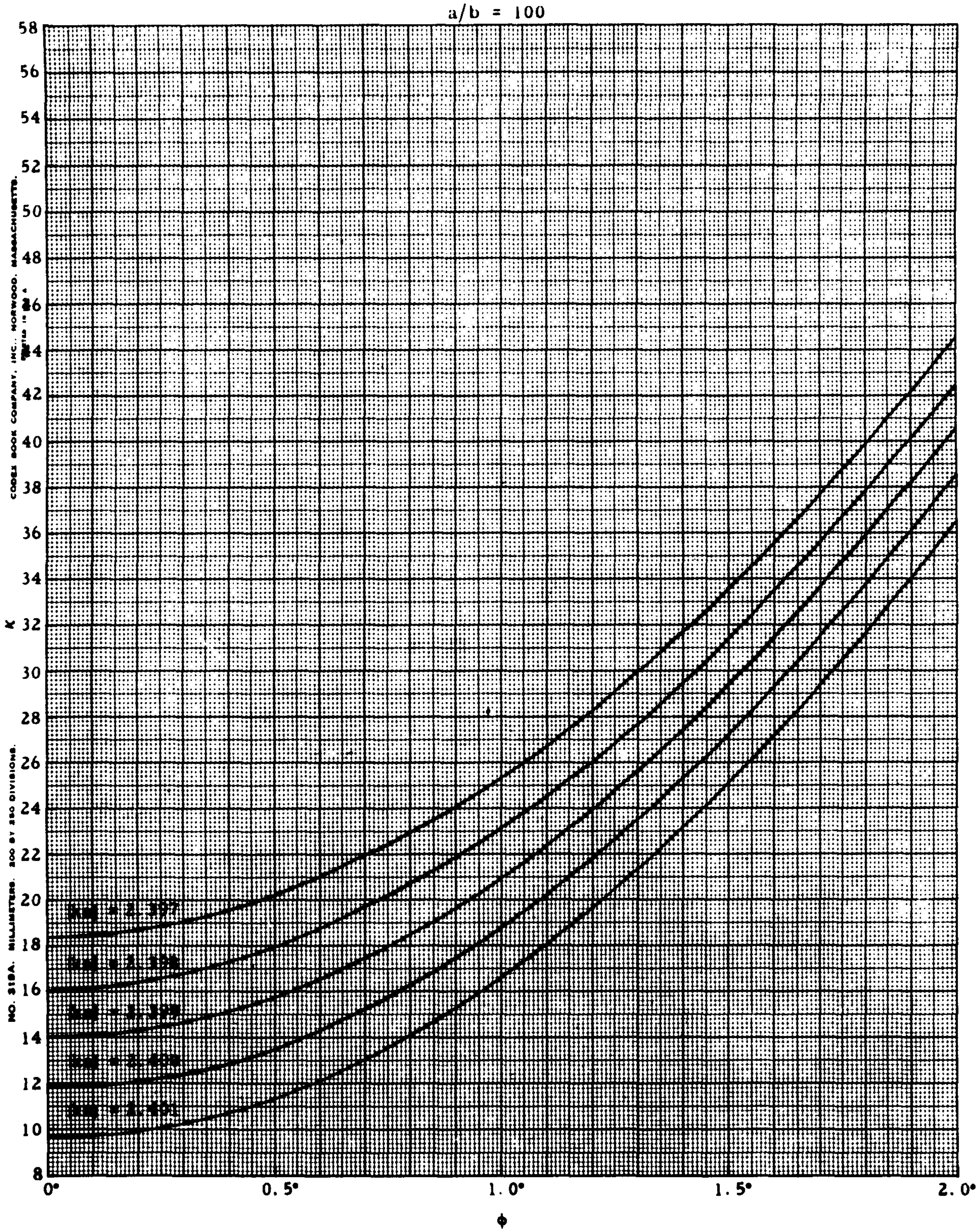

Fig. 8
κ vs ϕ for a/b = 100

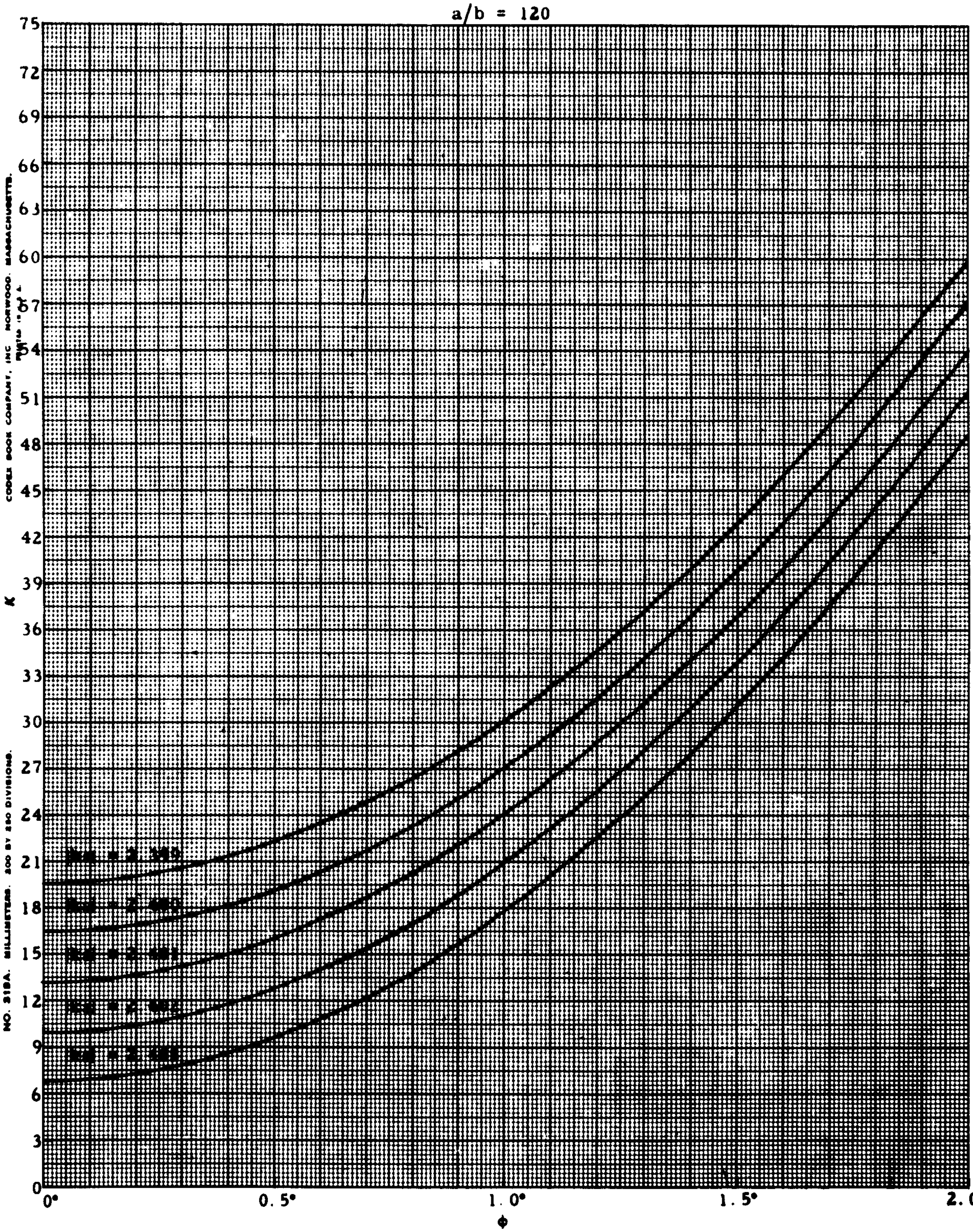

Fig. 9
κ vs ϕ for a/b = 120

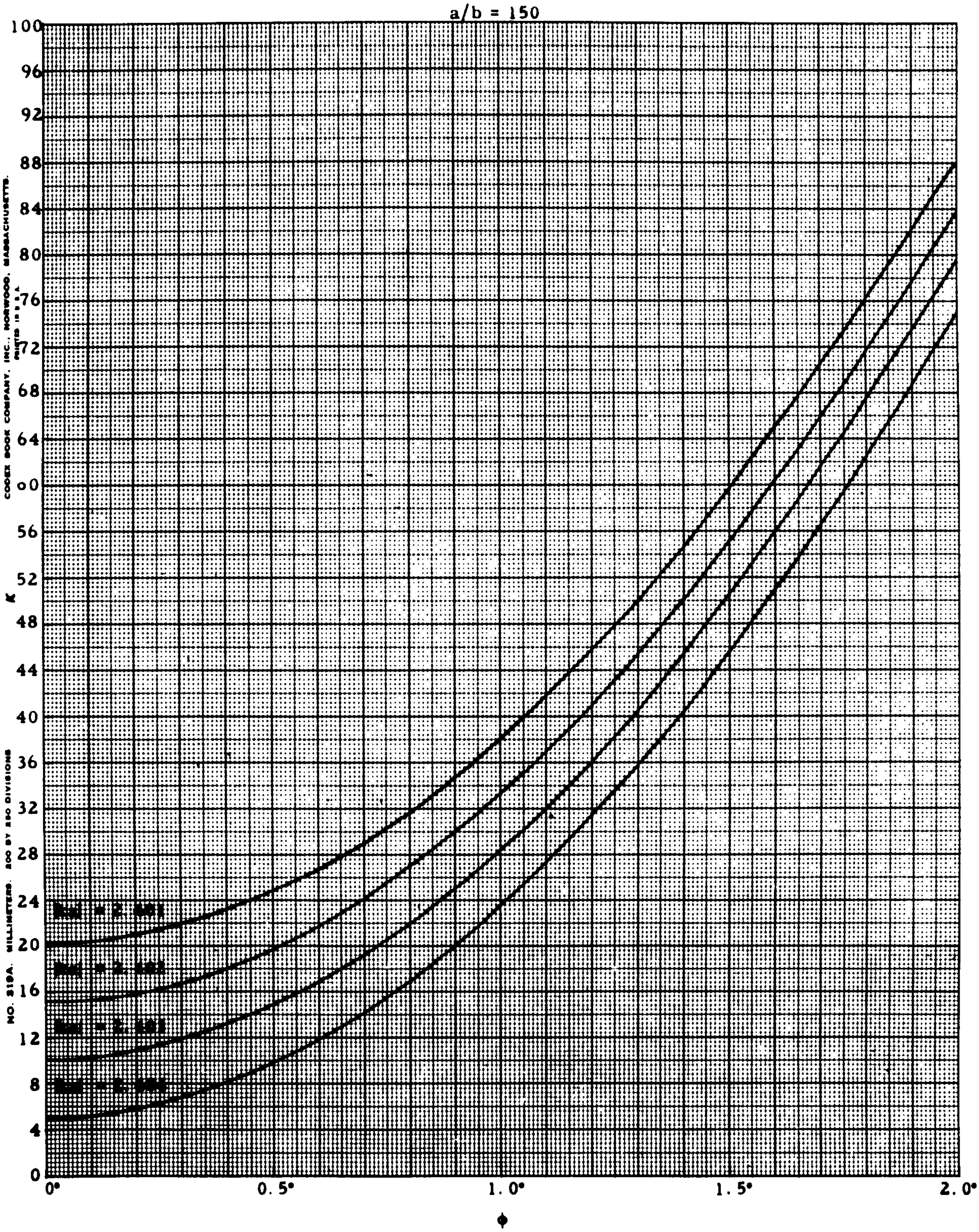

Fig. 10
κ vs ϕ for a/b = 150

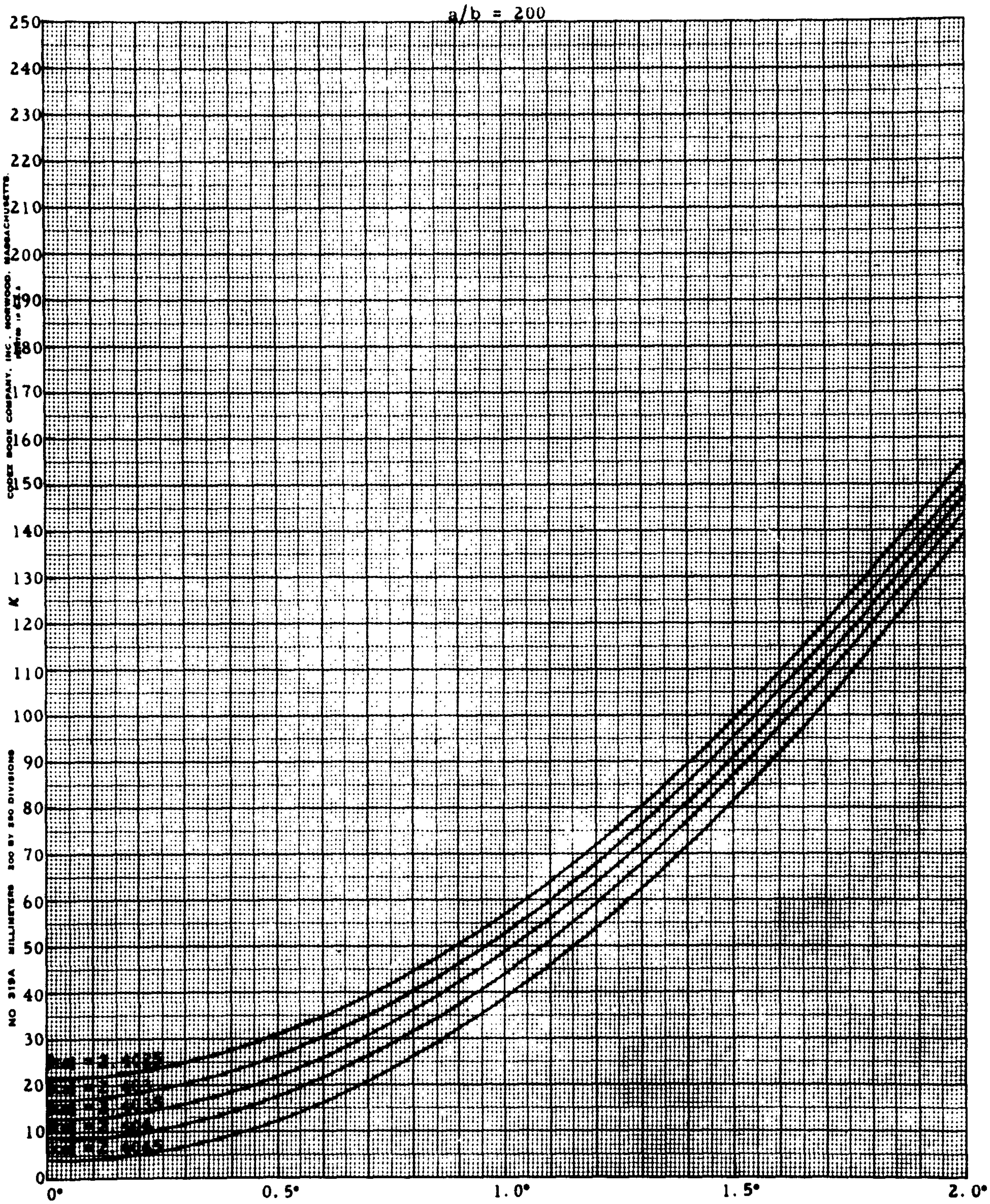

Fig. 11
κ vs ϕ for a/b = 200

a/b = 67

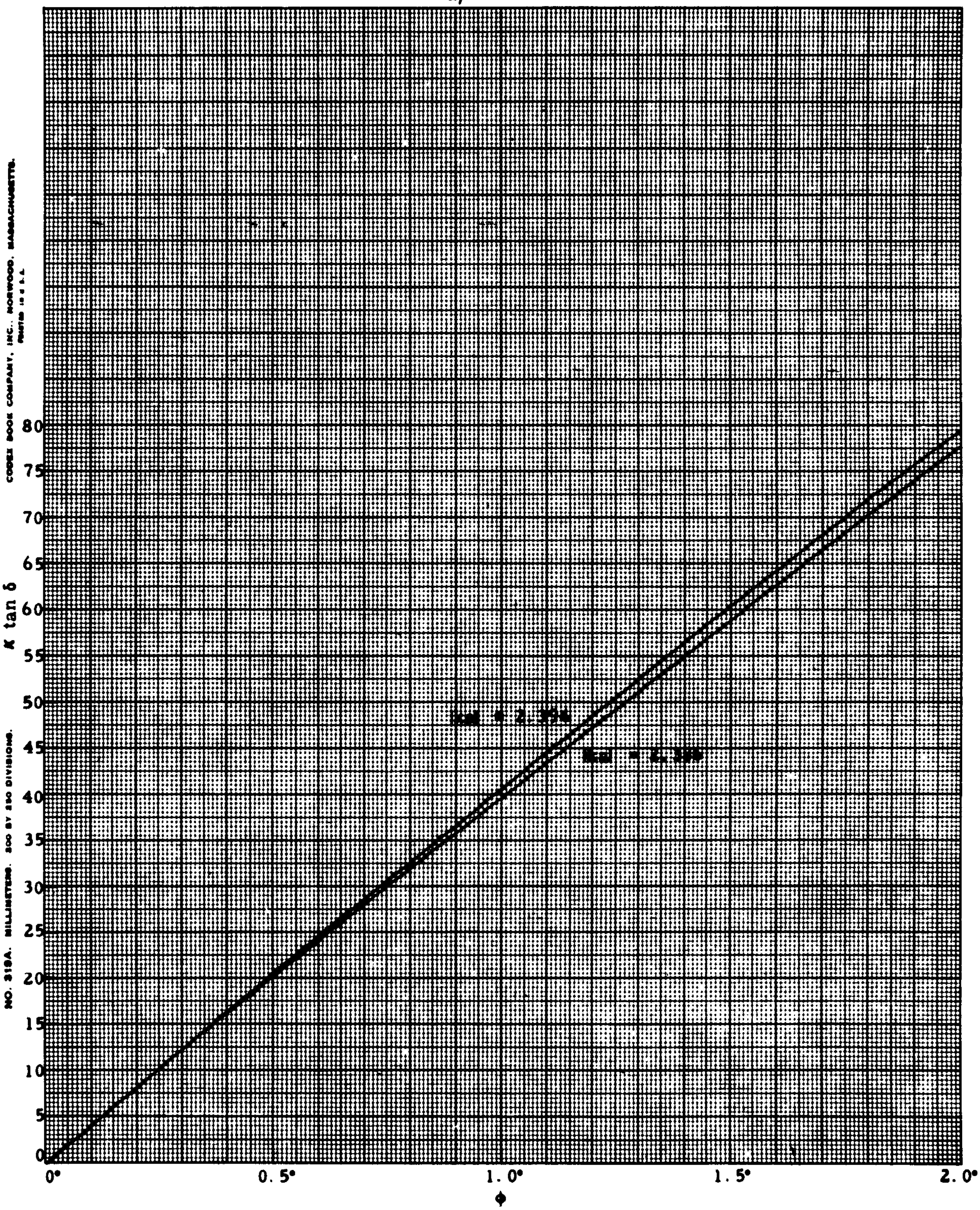

Fig. 12
$\kappa \tan \delta \cos \phi$ for a/b = 67

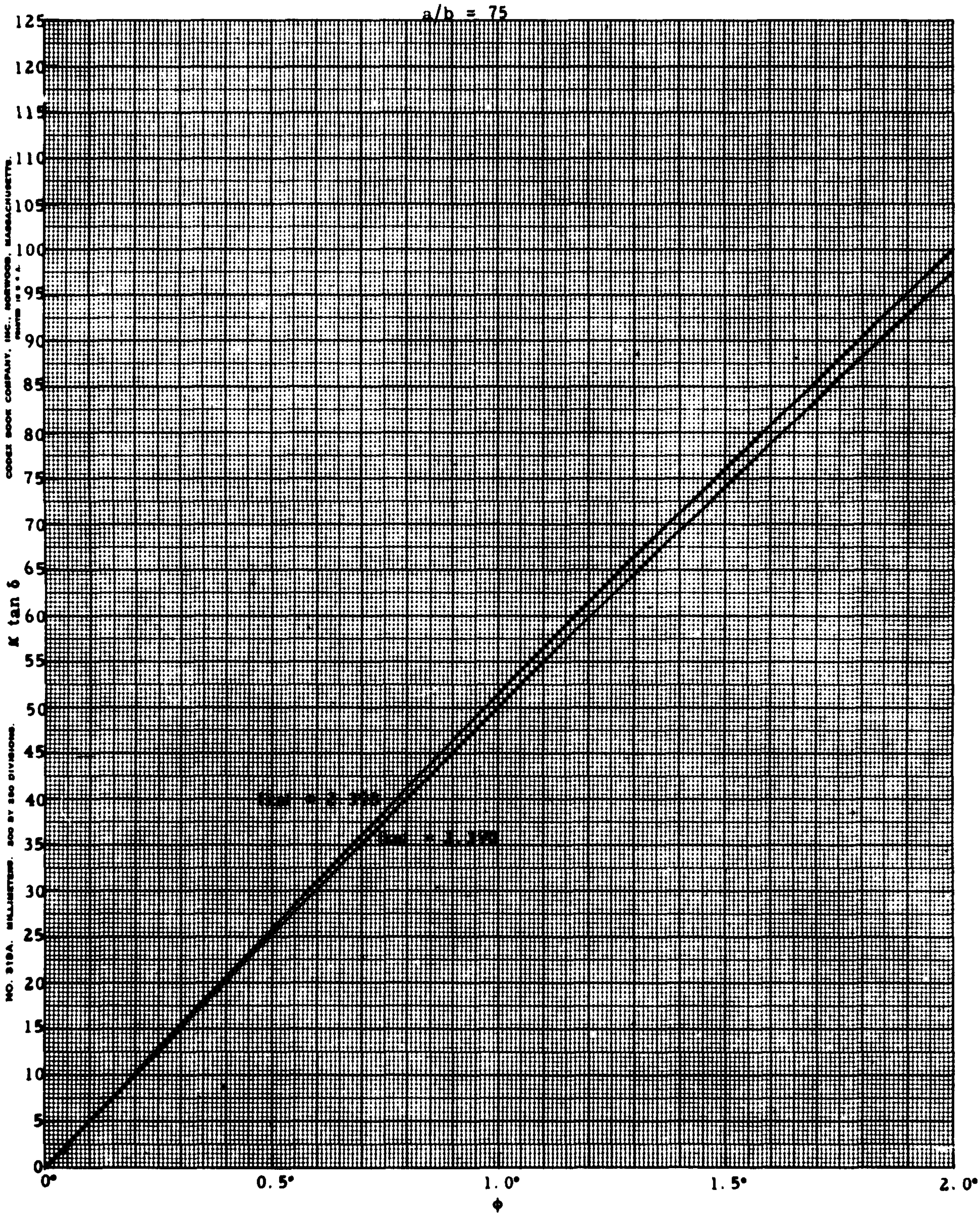

Fig. 13
$\kappa \tan \delta \cos \phi$ for a/b = 75

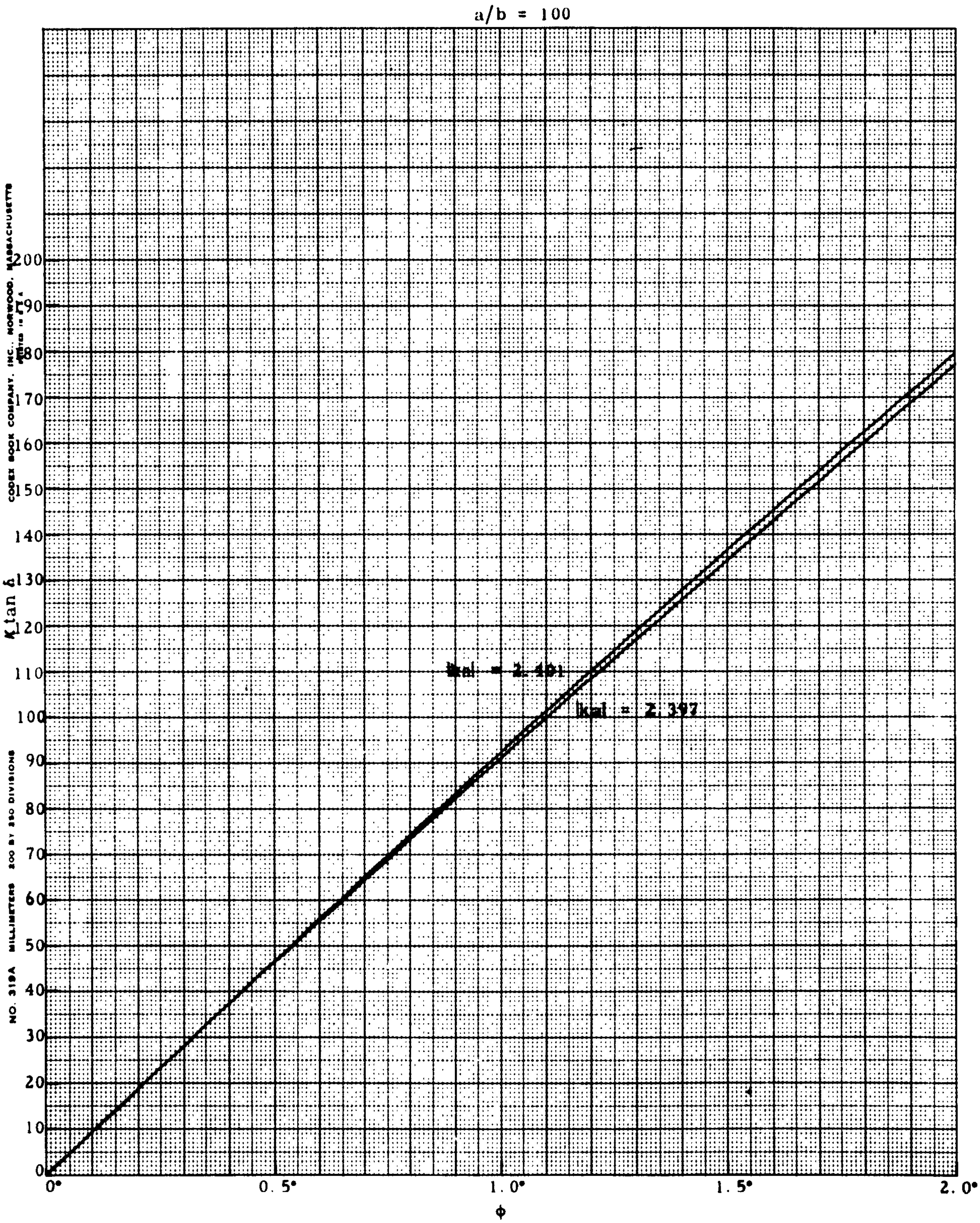

Fig. 14

$\kappa \tan \delta \cos \phi$ for a/b = 100

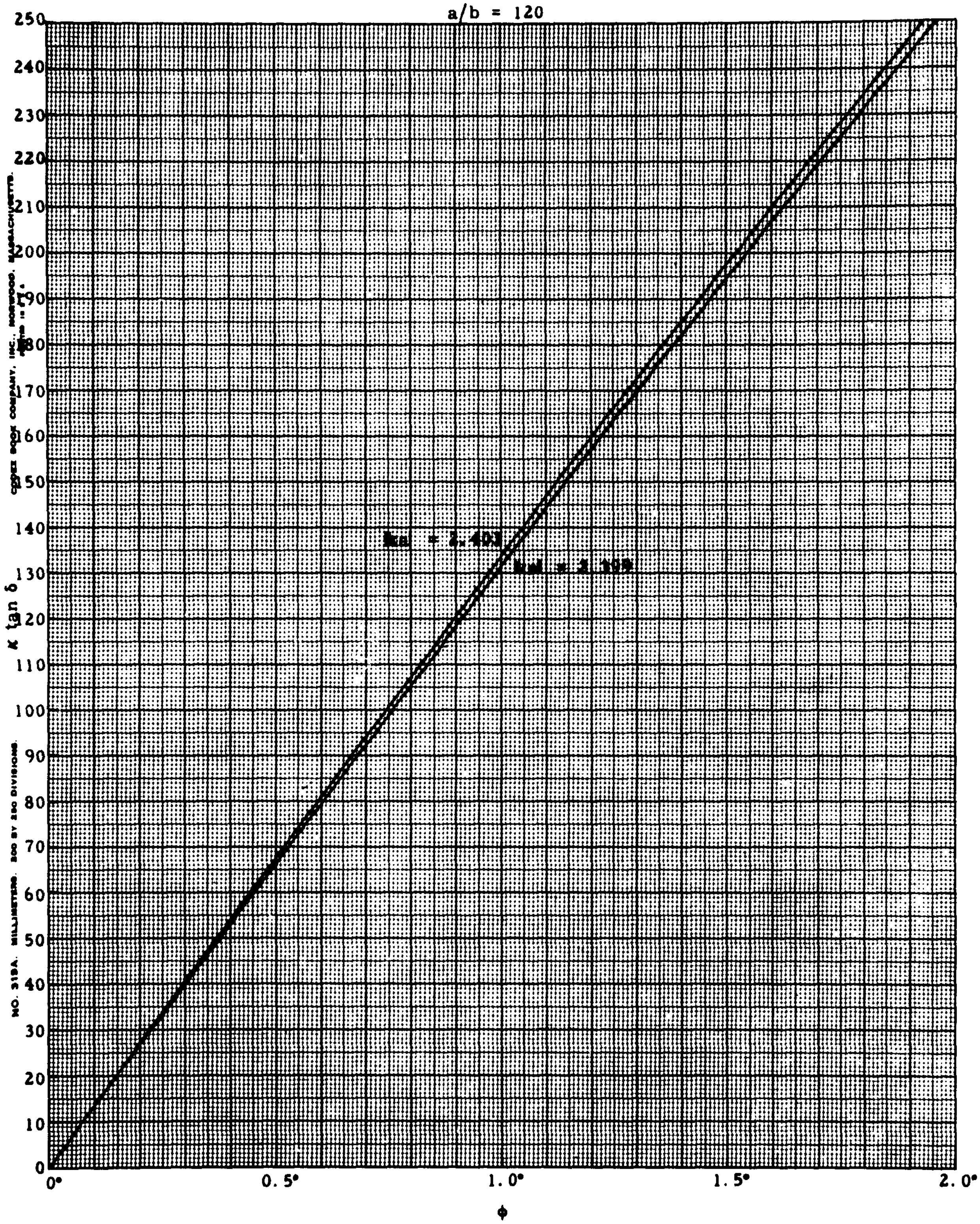

Fig. 15

$\kappa \tan \delta \cos \phi$ for a/b = 120

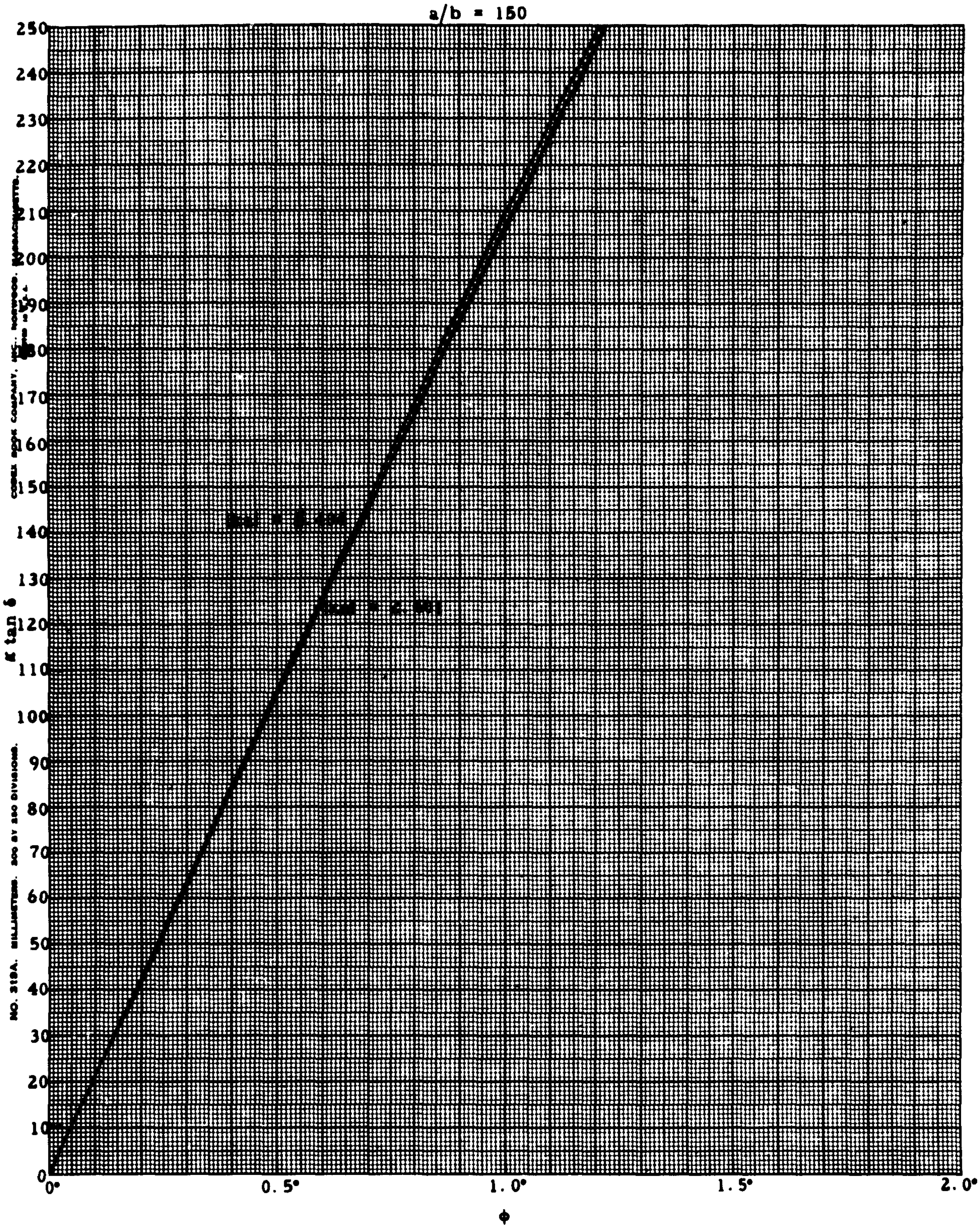

Fig. 16

$\kappa \tan \delta \cos \phi$ for a/b = 150

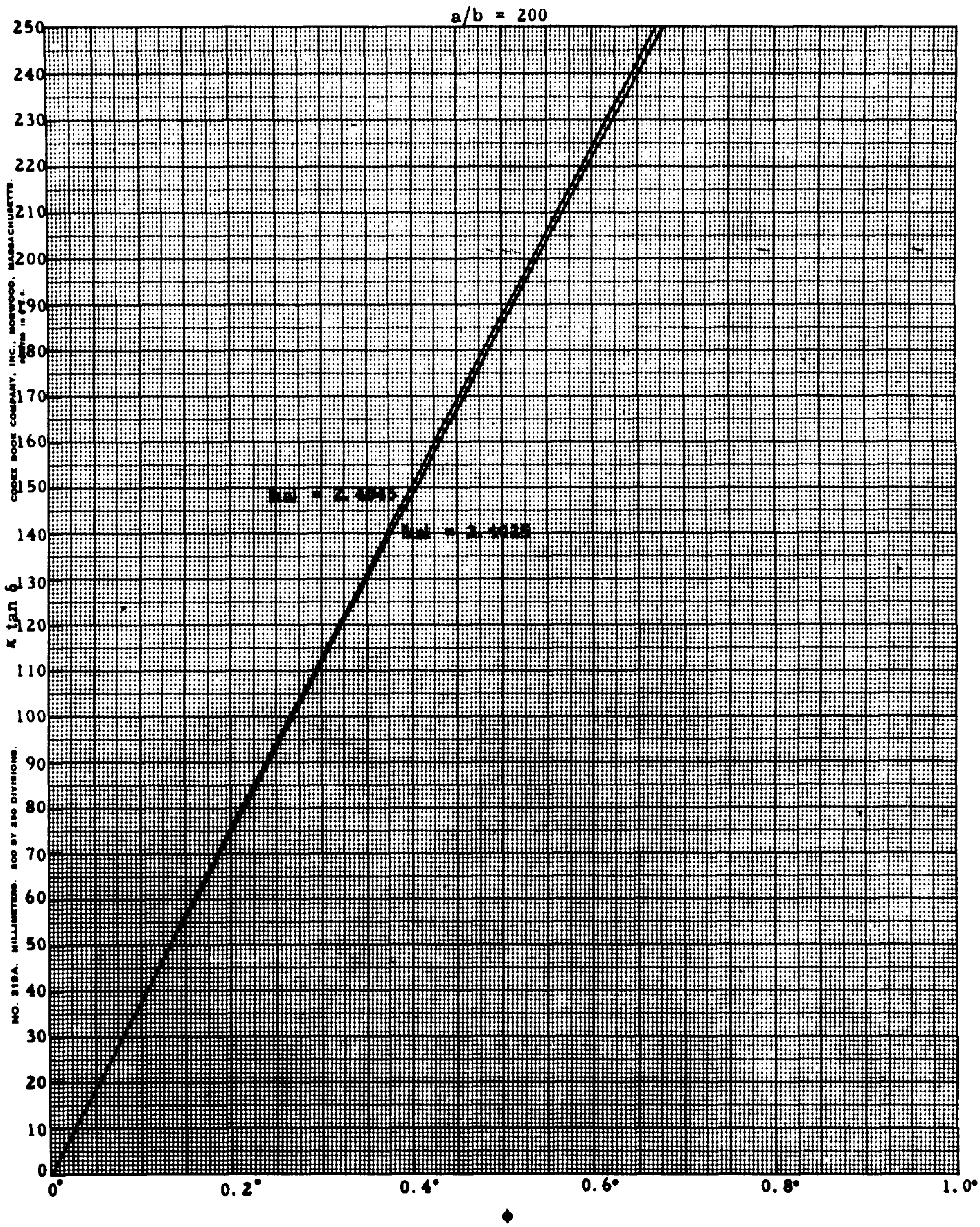

Fig. 17
$\kappa \tan \delta \cos \phi$ for $a/b = 200$

The electric field just outside the sphere is $(E_r^2 + E_\theta^2 + E_\phi^2)^{1/2}$. Integration of the fields for an infinitesimal expansion of the sphere from b to b + db gives

$$2b^2 db + \frac{4(\epsilon_c - \epsilon_0)^2}{(2\epsilon_0 + \epsilon_c)^2} b^2 db .$$

Using the perturbation method

$$\frac{\omega_0^2 - \omega^2}{\omega_0^2} = \int E^2 dv = 4\pi b^3 E_0^2 \frac{6\epsilon_0^2 + 3\epsilon_c^2}{(2\epsilon_0 + \epsilon_c)^2}$$

$$= 4\pi b^3 E_0^2 \left[1 + \frac{2}{\kappa^2(1 - j \tan \delta)^2} + \ldots\right] \left[1 - \frac{4}{\kappa(1 - j \tan \delta)} + \ldots\right] .$$

The frequency shift can be written as:

$$\frac{\Delta\omega}{\omega_0} = 2\pi E_0 b^3 \left[1 - \frac{4}{\kappa^2(1 + \tan^2 \delta)}\right] \tag{27}$$

where $\Delta\omega = \omega_0 - \omega$. For spherical samples of high dielectric constant, the second term is negligible with respect to unity. Consequently, it is not a good method to determine the dielectric constant by means of the frequency shift, which is only determined by the dimension of the sphere.

The change of Q can be written as

$$\frac{1}{Q_{loaded}} - \frac{1}{Q} = \frac{16\pi E_0^2 b^3}{\kappa^2 \tan \delta} . \tag{28}$$

If the dielectric constant is known by means of an independent measurement, this can serve as a cross check for the value of conductivity. It may be pointed out here that for substances with dielectric constants of the order of, or less than, unity, the frequency shift is sensitive to the value of κ.

Samples of Disc Shape in a Cylindrical Cavity

This method has been used by E. Ledinegg and E. Fehrer[8] in measuring low-loss dielectric substances. If a disc shaped sample of thickness h is inserted in a cylindrical cavity of height ℓ, Maxwell's equations can then be solved for regions I, II, and III and the resonant frequency determined by the continuity condition at $z = z_0$

and $z = z_0 + h$. The continuity condition gives:

$$\epsilon_c \frac{\gamma_1}{\gamma_2} \tan \gamma_1 (z_0 + h - \ell) = \frac{\epsilon_c \frac{\gamma_1}{\gamma_2} \tan \gamma_1 z_0 + \tan \gamma_2 h}{1 - \epsilon_c \frac{\gamma_1}{\gamma_2} \tan \gamma_1 z_0 \tan \gamma_2 h} \tag{29}$$

where

$$\gamma_1 = \sqrt{k^2 - k_0^2}$$

$$\gamma_2 = \sqrt{\kappa(1 - j \tan \delta)\, k^2 - k_0^2}$$

If the argument of the tangent is small enough so that the functions can be replaced by their arguments, then Eq. (28) can be simplified to the form:

$$\frac{\Delta k}{k} = \left(1 - \frac{1}{\epsilon_c}\right) \frac{h}{2(\ell - z_0)} \tag{30}$$

Eq. (29) is the same as that obtained by the perturbation method. As the surface is perpendicular to the electric field, the frequency shift is not sensitive to the dielectric constant. Separating the real and imaginary part of ϵ and k

$$\frac{\Delta k_r}{k} = 1 - \frac{\epsilon_r}{(\epsilon_r + \epsilon_j)^2} \frac{h}{2(\ell - x)}$$

$$\frac{1}{Q_1} - \frac{1}{Q_0} = \frac{\epsilon_j h}{(\ell - x)} .$$

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